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Spherical Convergence, Bounded Variation, and Singular Integrals on the N-Torus

VICTOR L. SHAPIRO*

Department of Mathematics, University of California, Riverside, California 92502

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1. INTRODUCTION

We shall operate in N-dimensional Euclidean space, E_N, $N \geqslant 2$, and use the following notation:

$$\begin{aligned} x &= (x_1, \ldots, x_N), \\ (x, y) &= x_1 y_1 + \cdots + x_N y_N, \\ |x|^2 &= (x, x), \\ T_N &= \{x : -\pi \leqslant x_j < \pi, j = 1, \ldots, N\}, \end{aligned}$$

and

$$\bar{Z} = \text{the closure of } Z.$$

We shall say that f is of bounded variation on the N-dimensional torus, T_N, if

f is a finite-valued function defined in E_N which is periodic of period 2π in each variable, (1.1)

and if

there exists a finite constant γ such that for every partition of $\bar{T}_N$ into a finite union of nonoverlapping intervals $\{I_k\}_{k=1}^{n}$, (1.2)

$$\sum_{k=1}^{n} |f(I_k)| \leqslant \gamma.$$

By an interval we mean a closed set of the form

$$I = \{x : a_j \leqslant x_j \leqslant a_j + h_j, j = 1, \ldots, N\},$$

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and by $f(I)$ we mean the following:

$$\begin{aligned} f(I) = {} & f(a_1 + h_1, ..., a_N + h_N) - f(a_1, a_2 + h_2, ..., a_N + h_N) \\ & - f(a_1 + h_1, a_2, a_3 + h_3, ...) - \cdots \\ & - f(a_1 + h_1, a_2 + h_2, ..., a_{N-1} + h_{N-1}, a_N) \\ & + f(a_1, a_2, a_3 + h_3, ..., a_N + h_N) + \cdots \\ & + f(a_1 + h_1, a_2 + h_2, ..., a_{N-1}, a_N) + \cdots + (-1)^N f(a_1, ..., a_N). \end{aligned} \tag{1.3}$$

In particular, for $N = 2$,

$$f(I) = f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2) - f(a_1 + h_1, a_2) + f(a_1, a_2),$$

and for $N = 3$,

$$\begin{aligned} f(I) = {} & f(a_1 + h_1, a_2 + h_2, a_3 + h_3) - f(a_1, a_2 + h_2, a_3 + h_3) \\ & - f(a_1 + h_1, a_2, a_3 + h_3) - f(a_1 + h_1, a_2 + h_2, a_3) \\ & + f(a_1, a_2, a_3 + h_3) + f(a_1, a_2 + h_2, a_3) \\ & + f(a_1 + h_1, a_2, a_3) - f(a_1, a_2, a_3). \end{aligned}$$

(For a discussion of bounded variation in the plane, we refer the reader to [1]. For a discussion of formula (1.3) above, we refer the reader to [5, p. 79].)

Next, we introduce the singular kernel

$$K(x) = 2^N \Gamma(N)\, \pi^{N/2} x_1 \cdots x_N / |x|^{2N}\, \Gamma(N/2), \tag{1.4}$$

and, with m designating an integral lattice point, we define the periodic analog of K to be

$$K^*(x) = K(x) + \lim_{R\to\infty} \sum_{1\leqslant |m| \leqslant R} [K(x + 2\pi m) - K(2\pi m)]$$
$$\text{for} \quad x \text{ in } E_N - \bigcup_m \{2\pi m\}. \tag{1.5}$$

We observe that the series in (1.5) is uniformly convergent, both on T_N and on every compact subset of $D = E_N - \bigcup_m \{2\pi m\}$. Also, on D, $K^*(x)$ is a periodic function of period 2π in each variable.

For f in $L^1(T_N)$ we shall designate the m-th Fourier coefficient of f by $\hat{f}(m)$. Thus,

$$\hat{f}(m) = (2\pi)^{-N} \int_{T_N} f(x)\, e^{-i(m,x)}\, dx. \tag{1.6}$$

Now, $K^*(x)$ is not in $L^1(T_N)$. Nevertheless, its principal-valued m-th Fourier coefficient exists and we designate it also by $\hat{K}^*(m)$. So, in particular, with

$B(x, r)$ representing the N-ball with center x and radius r, we have, from (1.5), that

$$\hat{K}^*(m) = \lim_{\epsilon\to 0} (2\pi)^{-N} \int_{T_N - B(0,\epsilon)} K^*(x)\, e^{-i(m,x)}\, dx$$
$$= \lim_{\epsilon\to 0} \lim_{R\to\infty} (2\pi)^{-N} \int_{B(0,R)-B(0,\epsilon)} K(x)\, e^{-i(m,x)}\, dx,$$

and, consequently, from [8, p. 69], that

$$\hat{K}^*(m) = (-i)^N m_1 \cdots m_N/|m|^N, \qquad \text{for} \qquad m \neq 0, \tag{1.7}$$

and

$$\hat{K}^*(0) = 0.$$

For f in $L^1(T_N)$, we shall designate by $S[f]$ the Fourier series of f, and by $\tilde{S}[f]$ the "conjugate" series of f (see [4, p. 259] and [8, p. 41]) with respect to the kernel K^*. Thus,

$$\begin{aligned} S[f] &= \textstyle\sum \hat{f}(m)\, e^{i(m,x)}, \\ \tilde{S}[f] &= \textstyle\sum \hat{f}(m)\, \hat{K}^*(m)\, e^{i(m,x)}. \end{aligned} \tag{1.8}$$

We shall say that $\tilde{S}[f]$ is spherically convergent at x^0 if

$$\lim_{R\to\infty} \sum_{|m|\leqslant R} \hat{f}(m)\, \hat{K}^*(m)\, e^{i(m,x^0)} \qquad \text{exists and is finite.} \tag{1.9}$$

If this limit exists and is finite, we shall refer to it as the spherical sum of $\tilde{S}[f]$.

In this paper, we intend to establish the following result:

THEOREM. *Suppose that f is a finite-valued function on E_N which is periodic of period 2π in each variable, belongs to $L^1(T_N)$, and is of bounded variation on T_N. Then a necessary and sufficient condition for the spherical convergence of $\tilde{S}[f]$ at x^0 is the existence and finiteness of the limit*

$$\lim_{\epsilon\to 0} (2\pi)^{-N} \int_{T_N - B(0,\epsilon)} f(x^0 - x)\, K^*(x)\, dx \tag{1.10}$$

which represents then the spherical sum of $\tilde{S}[f]$ at x^0.

If $N = 1$, it follows from (1.4), (1.5) and [10; (21), p. 73] that $K^*(x) = \cot x_1/2$, and from (1.7) and [10, p. 3] that $\tilde{S}[f]$ is the classical conjugate series of f. Consequently, the theorem stated above is precisely the multi-dimensional analog of the classical theorem due to W. H. Young [9] as stated in [10, p. 59] and in [2, p. 55].

2. Fundamental Lemmas

Before proving our first lemma, we need some more notation. In particular, we set

$$H_0(x, t) = \sum_{m \neq 0} e^{i(m,x)-|m|t} \mid m \mid^{-2}, \qquad \text{for} \quad t > 0, \tag{2.1}$$

and observe, from [7, p. 72], that

$$\lim_{t \to 0} H_0(x, t) = H_0(x) \qquad \text{is finite in} \quad E_N - \bigcup_m \{2\pi m\}. \tag{2.2}$$

Next, we set

$$H(x, t) = (-1)^N \sum_{m \neq 0} e^{i(m,x)-|m|t} \mid m \mid^{-N}, \qquad \text{for} \quad t > 0, \tag{2.3}$$

and observe from (2.1) that

$$\partial^{N-2} H(x, t)/\partial t^{N-2} = H_0(x, t). \tag{2.4}$$

Consequently, we conclude from the Cauchy criterion, (2.2), and (2.4) that

$$\lim_{t \to 0} H(x, t) = H(x) \qquad \text{is finite in} \quad E_N - \bigcup_m \{2\pi m\}. \tag{2.5}$$

Also, it follows from the Riesz–Fischer theorem that

$$H(x) \text{ is in } L^2(T_N),$$
$$\hat{H}(m) = (-1)^N \mid m \mid^{-N} \qquad \text{for} \qquad m \neq 0, \tag{2.6}$$

and

$$\hat{H}(0) = 0.$$

We shall say that μ is a measure in $\mathbf{M}(T_N)$ if (i) μ is a countably additive set function defined on the bounded Borel sets of E_N (ii) μ is of finite total variation on T_N, and (iii) $\mu(B + 2\pi m) = \mu(B)$ for every bounded Borel set B and every integral lattice point m. The first lemma we prove is the following:

Lemma 1. *Suppose that f is a finite-valued function on E_N which is periodic of period 2π in each variable, belongs to $L^1(T_N)$, and is of bounded variation on T_N. Then there is a function g in $L^2(T_N)$ and there is a measure μ in $\mathbf{M}(T_N)$ such that the following two properties hold:*

$$g(x) = (2\pi)^{-N} \int_{T_N} H(x - y)\, d\mu(y) \quad \textit{almost everywhere in } E_N\,; \tag{2.7}$$

$$\tilde{S}[f] = S[g]. \tag{2.8}$$

To prove the lemma, we set for $t > 0$ and x in E_N,

$$f(x, t) = \sum_m \hat{f}(m)\, e^{i(m,x)-|m|t}, \tag{2.9}$$

and

$$P(x, t) = \sum_m e^{i(m,x)-|m|t}. \tag{2.10}$$

Now, it follows from the Poisson summation formula [3, pp. 30–32] that

$$2^{-N}\pi^{(1-N)/2}P(x, t) = \Gamma[(N + 1)/2]\, t \sum_m [t^2 + |\, 2\pi m + x\,|^2]^{-(N+1)/2}, \tag{2.11}$$

and, consequently, that

$$P(x, t) > 0 \qquad \text{for} \qquad t > 0 \qquad \text{and} \qquad x \text{ in } E_N. \tag{2.12}$$

Next, by $V[f, T_N]$ we designate the total variation of f on T_N, so that, in particular, $V[f, T_N]$ represents the sup of the sum in (1.2) taken over all partitions of $\bar{T}_N$ into a finite union of nonoverlapping intervals.

For an interval $I = \{x : a_j \leqslant x_j \leqslant a_j + h_j, j = 1,..., N\}$ we define $f(I, t)$ by the same formula as in (1.3), with $f(x)$ replaced by $f(x, t)$.

We, next, establish the following:

$$V[f(\cdot, t), T_N] \leqslant V[f, T_N] \qquad \text{for} \qquad t > 0. \tag{2.13}$$

To prove (2.13), observe from (2.9) and (2.10) that

$$f(x, t) = (2\pi)^{-N} \int_{T_N} f(x - y)\, P(y, t)\, dy. \tag{2.14}$$

Let $\{I_k\}_{k=1}^n$ be a partition of $\bar{T}_N$ into a finite union of nonoverlapping intervals. Then from (2.12) and (2.14) we have

$$|\, f(I_k, t)| \leqslant (2\pi)^{-N} \int_{T_N} |\, f(I_k - y)|\, P(y, t)\, dy$$

and, consequently,

$$\sum_{k=1}^{n} |\, f(I_k, t)| \leqslant (2\pi)^{-N} \int_{T_N} V(f, T_N)\, P(y, t)\, dy. \tag{2.15}$$

Relation (2.13) then follows from (2.15) and the fact that

$$(2\pi)^{-N} \int_{T_N} P(y, t)\, dy = 1.$$

Next, we observe that $f(x, t)$ is in $C^\infty(E_N)$ for $t > 0$, and, using the notation $\partial^N f(x, t)/\partial x_1 \cdots \partial x_N = f_{x_1 \cdots x_N}(x, t)$, we consequently have

$$f(I, t) = \int_I f_{x_1 \cdots x_N}(x, t)\, dx \tag{2.16}$$

for every interval I.

From (2.13) and (2.16), we obtain that

$$\sum_{k=1}^{n} \left| \int_{I_k} f_{x_1 \cdots x_N}(x, t)\, dx \right| \leqslant V(f, T_N) \qquad \text{for} \quad t > 0, \tag{2.17}$$

for every partition $\{I_k\}_{k=1}^{N}$ of T_N into a nonoverlapping union of intervals. But, then, it follows immediately from (2.17) that

$$\int_{T_N} |f_{x_1 \cdots x_N}(x, t)|\, dx \leqslant V(f, T_N) \qquad \text{for} \quad t > 0. \tag{2.18}$$

However, $C(T_N)$ is separable. Consequently, it follows from the notion of weak* convergence (see [6, pp. 258–9]) that there is a μ in $\mathbf{M}(T_N)$ and a sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k \to 0$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \int_{T_N} \lambda(x) f_{x_1 \cdots x_N}(x, t_k)\, dx = \int_{T_N} \lambda(x)\, d\mu(x) \tag{2.19}$$

for every function $\lambda(x)$ in $C(E_N)$, periodic of period 2π in each variable (so that $\lambda \in C(T_N)$.)

Choosing $\lambda(x) = e^{-i(m,x)}/(2\pi)^N$, we conclude from (2.9) and (2.19) that

$$\hat{\mu}(m) = (i)^N m_1 \cdots m_N \hat{f}(m), \tag{2.20}$$

where

$$\hat{\mu}(m) = (2\pi)^{-N} \int_{T_N} e^{-i(m,x)}\, d\mu(x).$$

It follows from Fubini's theorem and the periodicity of H and μ that

$$\int_{T_N} |H(x - y)|\, d\,|\mu| < \infty \quad \text{almost everywhere in } E_N, \tag{2.21}$$

where $|\mu|$ represents the total variation measure associated with μ. Setting $Z = \{x \in E_N : \int_{T_N} |H(x - y)|\, d\,|\mu|\,(y) = \infty\}$ and observing that Z is of Lebesgue measure zero, we define

$$\begin{aligned} g(x) &= (2\pi)^{-N} \int_{T_N} H(x - y)\, d\mu(y) \qquad && \text{for} \quad x \text{ in } E_N - Z, \\ &= 0 && \text{for} \quad x \text{ in } Z. \end{aligned} \tag{2.22}$$

It follows from Fubini's theorem and (2.6) that g is in $L^1(T_N)$, and from (2.6), (2.20), and (2.22) that

$$\hat{g}(m) = (-i)^N \hat{f}(m)\, m_1 \cdots m_N \mid m \mid^{-N} \qquad \text{for} \quad m \neq 0, \qquad \hat{g}(0) = 0. \tag{2.23}$$

But $\sum_{m \neq 0} \mid \hat{\mu}(m) \mid^2 \mid m \mid^{-2N} < \infty$. Consequently, it follows from (2.20), (2.23), and the Riesz–Fischer theorem that g is actually in $L^2(T_N)$. The proof of the lemma is, therefore, complete, for (2.22) establishes (2.7), and (1.7), (1.8), and (2.23) give (2.8).

Next, we prove the following

LEMMA 2. *With f and μ as in Lemma* 1, *and* $f(x, t) = \sum \hat{f}(m)\, e^{i(m,x) - \mid m \mid t}$, *the following facts hold: Set*

$$J(x, t, q_1) = \int_0^{q_1} \cdots \int_0^{q_{N-1}} \left[\int_{B(x, q_N)} f_{y_1 \cdots y_N}(y, t)\, dy \right] q_2 \cdots q_N\, dq_2 \cdots dq_N .$$

Then, for $t > 0$ *and* $0 < q_1 < 1$,

$$\mid J(x, t, q_1) \mid \leqslant V(f, T_N) \int_0^{q_1} \cdots \int_0^{q_{N-1}} q_2 \cdots q_N\, dq_2 \cdots dq_N , \tag{2.24}$$

and

$$\lim_{t \to 0} J(x, t, q_1) = \int_0^{q_1} \cdots \int_0^{q_{N-1}} \mu[B(x, q_N)]\, q_2 \cdots q_N\, dq_2 \cdots dq_N . \tag{2.25}$$

Relation (2.24) follows immediately from (2.18).

To establish (2.25), observe from (2.10) and (2.20) that

$$f_{y_1 \cdots y_N}(y, t) = (2\pi)^{-N} \int_{T_N} P(y - z, t)\, d\mu(z).$$

For q_N such that $0 < q_N < 1$, define $\mathscr{X}_{B(0, q_N)}(x)$ to be the characteristic function of $B(0, q_N)$, for x in T_N, and define it throughout the rest of E_N by the periodicity of period 2π in each variable. Letting $h(x, q_1)$ designate the right side of (2.25), and observing that

$$\begin{aligned} \int_{B(x, q_N)} & f_{y_1 \cdots y_N}(y, t)\, dy \\ &= \int_{T_N} \mathscr{X}_{B(0, q_N)}(y - x) f_{y_1 \cdots y_N}(y, t)\, dy \\ &= (2\pi)^{-N} \int_{T_N} \int_{T_N} \mathscr{X}_{B(0, q_N)}(y + z - x)\, P(y, t)\, d\mu(z)\, dy \\ &= (2\pi)^{-N} \int_{T_N} \mu[B(x - y, q_N)]\, P(y, t)\, dy, \end{aligned}$$

we conclude that

$$J(x, t, q_1) = (2\pi)^{-N} \int_{T_N} h(x - y, q_1)\, P(y, t)\, dy. \tag{2.26}$$

It is clear that $h(x, q_1)$ is a continuous periodic function on E_N, of period 2π in each variable. We conclude from [7, p. 56] that $\lim_{t\to 0} J(x, t, q_1) = h(x, q_1)$, which establishes (2.25) and completes the proof of the lemma.

The next lemma we prove is the following:

LEMMA 3. *Let $0 < r_1 < r_2 < \infty$ and let $c(r)$ be a continuous function on the closed interval $[r_1, r_2]$. Also, let f and μ be as in Lemma 1. Then, for x in E_N, the following identity holds:*

$$\int_{B(0,r_2)-B(0,r_1)} c(|y|)\, f(x - y)\, y_1 \cdots y_N\, |y|^{-2N}\, dy$$

$$= (-1)^N \int_{r_1}^{r_2} \int_0^{q_1} \cdots \int_0^{q_{N-1}} c(q_1)\, \mu[B(x, q_N)]\, q_1^{-2N} q_1 \cdots q_N dq_1 \cdots dq_N\,. \tag{2.27}$$

To prove the lemma, we let $f(x, t)$ be as in Lemma 2. Also, we designate the $(N - 1)$-sphere with center 0 and radius r by $S(0, r)$, and we let $dS(y)$ designate the natural $(N - 1)$-volume element on $S(0, r)$. Then, for $t > 0$, we have, with $J(x, t, q_1)$ as in Lemma 2,

$$\begin{aligned}
&(-1)^N \int_{S(0,q_1)} f(x - y, t)\, y_1 \cdots y_N\, dS(y) \\
&= \int_{S(0,q_1)} f(x + y, t)\, y_1 \cdots y_N\, dS(y) \\
&= q_1 \int_{B(0,q_1)} f_{y_1}(x + y, t)\, y_2 \cdots y_N\, dy \\
&= \int_0^{q_1} q_1\, dq_2 \int_{S(0,q_2)} f_{y_1}(x + y, t)\, y_2 \cdots y_N\, dS(y) \\
&\qquad \vdots \\
&= \int_0^{q_1} \cdots \int_0^{q_{N-1}} q_1 \cdots q_N \left[\int_{B(0,q_N)} f_{y_1 \cdots y_N}(x + y, t)\, dy\right] dq_2 \cdots dq_N \\
&= q_1 J(x, t, q_1).
\end{aligned} \tag{2.28}$$

Now, as is well-known, $f(y, t) \to f(y)$ in the L^1-norm on T_N, as $t \to 0$. (This fact follows almost immediately from (2.11) and (2.14).) Consequently, from (2.24), (2.25), and (2.28) we obtain that the left side of (2.27) is equal to

$$\lim_{t\to 0}\int_{B(0,r_2)-B(0,r_1)} c(|y|)\, f(x-y,t)\, y_1\cdots y_N\, |y|^{-2N}\, dy$$

$$= \lim_{t\to 0}\int_{r_1}^{r_2} c(q_1)\, q_1^{-2N}\left[\int_{S(0,q_1)} f(x-y,t)\, y_1\cdots y_N\, dS(y)\right] dq_1$$

$$= \lim_{t\to 0}\int_{r_1}^{r_2} c(q_1)\, q_1^{-2N}(-1)^N\, q_1 J(x,t,q_1)\, dq_1$$

$$= (-1)^N\int_{r_1}^{r_2}\int_0^{q_1}\cdots\int_0^{q_{N-1}} c(q_1)\, q_1^{-2N}\mu[B(x,q_N)]\, q_1\cdots q_N\, dq_1\cdots dq_N .$$

and the proof of the lemma is complete.

A measure μ in $\mathbf{M}(T_N)$ is said to be continuous at x if $\mu[B(x,r)]\to 0$ as $r\to 0$.

Next, we prove the following:

LEMMA 4. *Let f, g, and μ be as in Lemma 1. Set*

$$\tilde{f}(x:t) = (2\pi)^{-N}\int_{T_N-B(0,t)} f(x-y)\, K^*(y)\, dy, \tag{2.29}$$

and

$$g(x,t) = \sum_m \hat{f}(m)\, \hat{K}^*(m)\, e^{i(m,x)-|m|t}. \tag{2.30}$$

Then

$$\lim_{t\to 0}\sup|g(x,t)-\tilde{f}(x:t)| < \infty \qquad \textit{for} \quad x \textit{ in } T_N, \tag{2.31}$$

and

$$\lim_{t\to 0}|g(x,t)-\tilde{f}(x:t)| = 0 \qquad \textit{if} \quad \mu \textit{ is continuous at } x. \tag{2.32}$$

To establish Lemma 4, we first observe from (1.7) that

$$\lim_{t\to 0}\int_{T_N-B(0,t)} K^*(y)\, dy = 0.$$

Consequently, we can assume from the start that $\hat{f}(0) = 0$. But, then, from (2.29) and (1.5) we obtain that

$$\tilde{f}(x:t) = (2\pi)^{-N}\int_{T_N-B(0,t)} f(x-y)\, K(y)\, dy$$

$$+ \lim_{R\to\infty}\sum_{1\leq|m|\leq R}(2\pi)^{-N}\int_{T_N} f(x-y)\, K(y+2\pi m)\, dy + o(1).$$

Also, since

$$\lim_{R\to\infty}\Big[\int_{B(0,R)-T_N} f(x-y)\,K(y)\,dy - \sum_{1\leqslant|m|\leqslant R}\int_{T_N} f(x-y)\,K(y+2\pi m)\,dy\Big] = 0,$$

(see [8, p. 45]), we conclude that

$$\tilde{f}(x:t) = \lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,t)} f(x-y)\,K(y)\,dy + o(1). \tag{2.33}$$

We note, in particular, that the limit on the right side of (2.33) exists and is finite for $t > 0$.

Next, with $\beta = (N-2)/2$, we set

$$A_N^{\beta}(r) = \Gamma(N/2)\int_0^{\infty}[e^{-s/r}J_{\beta+N}(s)\,s^{\beta+1}/2^{N/2}\Gamma(N)]\,ds \tag{2.34}$$

and observe from [8, pp. 64–67] and (2.30) that

$$g(x,t) = \lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)} f(x-y)\,A_N^{\beta}(|\,y\,|\,t^{-1})\,K(y)\,dy. \tag{2.35}$$

From [8, p. 64] we have

$$|\,A_N^{\beta}(|\,y\,|\,t^{-1})| \leqslant \Gamma(N)(|\,y\,|\,t^{-1})^N. \tag{2.36}$$

Consequently, it follows from (2.27) in Lemma 3, and (2.36) that

$$\begin{aligned}
&\Big|\int_{B(0,t)} f(x-y)\,A_N^{\beta}(|\,y\,|\,t^{-1})\,K(y)\,dy\Big| \\
&\leqslant \eta_N\int_0^t\int_0^{q_1}\cdots\int_0^{q_{N-1}}\{(q_1t^{-1})^N\,|\,\mu[B(x,q_N)]\,|\,q_1^{-2N}q_1\cdots q_N\}\,dq_1\cdots dq_N \\
&\leqslant \eta_N t^{-N}\int_0^t\int_0^{q_1}\cdots\int_0^{q_{N-1}}|\,\mu[B(x,q_N)]\,|\,dq_1\cdots dq_N \\
&\leqslant \eta_N\sup_{0<q_N\leqslant t}|\,\mu[B(x,q_N)]\,|,
\end{aligned}$$

where η_N is a constant depending on N.

We conclude from this last computation that

$$\limsup_{t\to 0}\Big|\int_{B(0,t)} f(x-y)\,A_N^{\beta}(|\,y\,|\,t^{-1})\,K(y)\,dy\Big| \tag{2.37}$$

is finite for every x in T_N and is zero in case μ is continuous at x.

Next, we observe from [8, Lemma 16] that

$$| A_N^\beta(r) - 1 | \leqslant \eta_N' r^{-1/2} \qquad \text{for} \qquad r > 1, \tag{2.38}$$

where η_N' is a constant depending on N.

Consequently, we obtain from (2.27) and (2.38) that, for $\delta > t$,

$$\begin{aligned}
&\left| \int_{B(0,\delta)-B(0,t)} f(x-y)[A_N^\beta(| y | t^{-1}) - 1]\, K(y)\, dy \right| \\
&\quad \leqslant \eta_N'' \int_t^\delta \int_0^{q_1} \cdots \int_0^{q_{N-1}} \{(tq_1^{-1})^{1/2}\, | \mu[B(x, q_N)] |\, q_1^{-2N} q_1 \cdots q_N\}\, dq_1 \cdots dq_N \\
&\quad \leqslant \eta_N'' \sup_{0<q_N<\delta} | \mu[B(x, q_N)] |\, t^{1/2} \int_t^\infty q_1^{-3/2}\, dq \\
&\quad \leqslant 2\eta_N'' \sup_{0<q_N<\delta} | \mu[B(x, q_N)] |,
\end{aligned} \tag{2.39}$$

where η_N'' is a constant depending on N but not on δ or t.

We observe that, for $\delta > 0$,

$$\lim_{R\to\infty} \int_{B(0,R)-B(0,\delta)} | f(x-y)\, K(y) |\, | y |^{1/2}\, dy < \infty. \tag{2.40}$$

Furthermore, from (2.38) we have, for $\delta > t$,

$$\begin{aligned}
&\left| \int_{B(0,R)-B(0,\delta)} f(x-y)[A_N^\beta(| y | t^{-1}) - 1]\, K(| y |)\, dy \right| \\
&\qquad \leqslant \eta_N' t^{1/2} \int_{B(0,R)-B(0,\delta)} | f(x-y)\, K(y) |\, | y |^{-1/2}\, dy.
\end{aligned} \tag{2.41}$$

We therefore conclude from (2.40) and (2.41) that

$$\lim_{t\to 0} \lim_{R\to\infty} \int_{B(0,R)-B(0,\delta)} f(x-y)[A_N^\beta(| y | t^{-1}) - 1]\, K(y)\, dy = 0. \tag{2.42}$$

From (2.39) and (2.42), we obtain

$$\limsup_{t\to 0} \lim_{R\to\infty} \left| \int_{B(0,R)-B(0,t)} \{f(x-y)[A_N^\beta(y)\, t^{-1} - 1]\, K(y)\, dy\} \right| \tag{2.43}$$

is finite for every x in T_N and is zero in case μ is continuous at x.

The desired conclusions of the lemma follow from (2.33), (2.35), (2.37), and (2.43).

Finally, we establish the following:

Remark. Let $a_m = 0(|m|^{-N})$ as $|m| \to \infty$ and suppose

$$A = \lim_{t \to 0} \sum_m a_m e^{-|m|t}$$

exists and is finite. Then $\sum a_m$ is spherically convergent to A.

To establish this statement, we set $b_j = \sum_{j \leqslant |m| < j+1} a_m$ and observe that $|b_j| \leqslant \sum_{j \leqslant |m| < j+1} |a_m| = 0(j^{-1}) = 0(j^{-1})$ as $j \to \infty$. Consequently, the remark will follow from the Littlewood Tauberian Theorem [10, p. 81] once we show that

$$\lim_{t \to 0} \left[\sum_{j=0}^{\infty} b_j e^{-jt} - \sum_m a_m e^{-|m|t} \right] = 0. \tag{2.44}$$

We observe that the absolute value of the expression in brackets in (2.44) is majorized by

$$\sum_{j=1}^{\infty} \sum_{j \leqslant |m| < j+1} |a_m| \, | e^{-jt} - e^{-|m|t} | \leqslant [1 - e^{-t}] \sum_{j=1}^{\infty} O(j^{-1}) \, e^{-jt}. \tag{2.45}$$

Since the right side of the inequality is $o(1)$ as $t \to 0$, (2.44) is established and the proof of our remark is complete.

3. Proof of the Theorem

Let μ be as in Lemma 1, and let $\tilde{f}(x : t)$ be as in (2.29). Since $\lim_{r \to 0} \mu[B(x^0, r)]$ exists and is finite, it follows from Lemma 3, with $c(r) = 1$ in the closed interval [0, 1], that

$$\limsup_{t \to 0} |\tilde{f}(x^0 : t)| < \infty \quad \text{implies that } \mu \text{ is continuous at } x^0. \tag{3.1}$$

If $\tilde{S}[f]$ is spherically convergent at the point x^0 to the finite value α, then it follows that

$$\lim_{t \to 0} g(x^0, t) = \alpha, \tag{3.2}$$

where $g(x, t)$ is defined in (2.30). Thus we obtain from (2.31) that $\limsup_{t \to 0} |\tilde{f}(x^0 : t)| < \infty$, and, consequently, from (3.1), that μ is continuous at x^0. Using (2.32), we get $\lim_{t \to 0} [g(x^0, t) - \tilde{f}(x^0 : t)] = 0$ and, therefore, from (3.2), $\lim_{t \to 0} \tilde{f}(x^0, t) = \alpha$. The necessity of the condition of the theorem is therefore established.

To prove sufficiency, suppose that

$$\lim_{t \to 0} \tilde{f}(x^0 : t) = \alpha, \qquad \text{where} \quad \alpha \text{ is finite}. \tag{3.3}$$

It follows from (3.1) that μ is continuous at x^0; consequently, from (2.32) and (3.3),

$$\lim_{t\to 0} g(x^0, t) = \alpha, \tag{3.4}$$

where $g(x^0, t) = \sum_m \hat{f}(m)\, \hat{K}^*(m)\, e^{i(m,x^0)} e^{-|m|t}$.

From Lemma 1 we obtain that

$$\hat{f}(m)\, \hat{K}^*(m) = (-1)^N\, \hat{\mu}(m) \mid m \mid^{-N}, \qquad m \neq 0, \tag{3.5}$$

where $\hat{\mu}(m)$ is defined in (2.20). Consequently, if we set

$$a_m = \hat{f}(m)\, \hat{K}^*(m)\, e^{i(m,x^0)}, \tag{3.6}$$

we have from (3.4), (3.5), and (3.6) that

$$\lim_{t\to 0} \sum_m a_m e^{-|m|t} = \alpha, \tag{3.7}$$

and from (3.5), (3.6), and (2.20) that

$$a_m = 0(\mid m \mid^{-N}) \qquad \text{as} \qquad \mid m \mid \to \infty. \tag{3.8}$$

But then it follows from the Remark, (3.7), and (3.8) that

$$\lim_{R\to\infty} \sum_{|m|\leqslant R} a_m = \alpha.$$

The last relation, in conjunction with (3.6) and (1.8), implies that $\tilde{S}[f]$ is spherically convergent at the point x^0 to the value α, establishing the sufficiency of the condition of the theorem; the proof of the theorem is complete.

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